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BLOW UP NEAR HIGHER MODES OF NONLINEAR WAVE EQUATIONS

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ABSTRACT. This paper is concerned with the instability properties of higher modes of the nonlinear wave equation $u_{tt} - \Delta u - f(u) = 0$ defined on a smoothly bounded domain with Dirichlet boundary conditions. It is shown that they are unstable in the sense that in any neighborhood of a higher mode there exists a solution of the given equation which blows up in finite time.

Introduction. The present paper is concerned with the instability properties of steady states of the nonlinear wave equation $u_{tt} - \Delta u - f(u) = 0$, defined on a smoothly bounded domain $D \subset \mathbf{R}^n$ with Dirichlet boundary conditions. The steady states are solutions of the elliptic equations $-\Delta u - f(u) = 0$ on D with Dirichlet boundary conditions.

A number of authors have studied conditions on the initial data and the nonlinearity f for which the solutions of the given wave equation blow up in finite time (see [**13**] for references). A blow up result for the positive solution of the elliptic equation (the ground state) was given by L. E. Payne and D. H. Sattinger in [**13**]. However, the instability of higher modes of nonlinear wave equations remained an open question. In this paper we show that for a general class of nonlinearities f in any neighborhood of a steady state we can find such initial data for which the solution of the given wave equation blows up in finite time.

In [**13**] L. E. Payne and D. H. Sattinger were able to find an invariant region such that any solution of the nonlinear wave equation with initial data in this region blows up in finite time. Any attempt to find such an invariant region for a higher mode seems to fail. However, we can approximate higher modes by initial data of solutions which blow up in finite time.

The existence of infinitely many solutions of elliptic boundary value problems on a bounded domain was studied in a number of papers (see, for example, [**2**, **3**, **5**, **7**, **8**, **14**]). We use the approach, given by A. Ambrosetti and P. H. Rabinowitz in [**3**], which is based on an application of the Ljusternik-Schnirelman category theory. The main idea is to consider the solutions of the elliptic equation as critical points of a functional. In our case, they are critical points of the restriction of a functional to an appropriate infinite-dimensional C^2-hypersurface, and therefore we can use the Ljusternik-Schnirelman category theory on Finsler manifolds (see [**11**]). The critical values are obtained using a minimum-maximum principle over sets of various genuses. The corresponding critical points are the steady states.

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We prove the following:

THEOREM. *Let b_k be the critical value which is obtained using the minimum-maximum principle over sets of genus $k \geq 2$. Then there exists an unstable steady state which corresponds to b_k. It is unstable in the sense that in any neighborhood of this steady state exists a solution of the given wave equation which blows up in finite time.*

In the following we employ the following notation: $\| \ \|$ is the norm in $H_0^1(D)$, defined by

$$\|u\|^2 = \int_D |\nabla u|^2 \, dx,$$

$\| \ \|_q$ is the usual norm in $L^q(D)$; by $\rightharpoonup$ we denote weak convergence; and by $\to$ strong convergence.

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1. Preliminaries. Consider the nonlinear wave equation

$$u_{tt} - \Delta u - f(u) = 0 \tag{1}$$

on a smoothly bounded domain $D \subset \mathbf{R}^n$ with the Dirichlet boundary condition $u_{|\partial D} = 0$.

The steady states of this equation are the nontrivial solutions of the elliptic equation

$$-\Delta u - f(u) = 0 \tag{2}$$

on D with the boundary condition $u_{|\partial D} = 0$.

Throughout this paper we will make the following assumptions:

(i) $F(z) = \int_0^z f(s)\, ds$;

(ii) $f \in C^2(\mathbf{R}, \mathbf{R}),\ f(0) = f'(0) = 0$;

(iii) f is strictly convex for $u > 0$;

(iv) f is odd;

(v) $uf''(u) = O(|u|^{q-2})$;

(vi) $(p+1)|F(z)| \leq zf(z) \leq \varepsilon |z|^2 + B|z|^q$, for all $z \in \mathbf{R}$, where $2 < p+1 \leq q \leq 2n/(n-2),\ \varepsilon, B > 0$.

For example $f(u) = |u|^2 u$.

Note that by (ii) and (iii) ε is arbitrarily small.

The above assumptions imply that $zf(z) > 0$, and $z(zf'(z) - f(z)) > 0$ for all $z \in \mathbf{R} - \{0\}$.

Consider the functionals $J, R\colon H_0^1(D) \to \mathbf{R}$, defined by

$$J(u) = \frac{1}{2}\int_D |\nabla u|^2 \, dx - \int_D F(u)\, dx$$

and

$$R(u) = \int_D |\nabla u|^2 \, dx - \int_D uf(u)\, dx.$$

Let

$$\tilde{M} = \{u \in H_0^1(D) | u \neq 0,\ R(u) \leq 0\} \quad \text{and} \quad M = \{u \in \tilde{M} | R(u) = 0\}.$$

The following proposition states some of the properties of M, J, and R that will be useful in the future.

PROPOSITION 1.1. (i) *$\tilde{M}$ is bounded away from zero.*

(ii) *There exists a $K > 0$ such that*

$$K\|u\| \leq J(u) - \tfrac{1}{2}R(u) < \infty$$

for all $u \in \tilde{M}$. In particular,

$$K\|u\| \leq J(u) < \infty$$

for all $u \in M$.

(iii) *The set M is a C^2-submanifold of $H_0^1(D)$ of codimension 1 which contains all the nontrivial equilibrium solutions of (1) in $H_0^1(D)$.*

(iv) *For any $u \in M$, the function $j\colon \mathbf{R}_+ \to \mathbf{R}$, defined by $j(\alpha) = J(\alpha u)$, achieves its maximum at $\alpha = 1$.*

(v) *For any $u \in \tilde{M}$ there exists an $\varepsilon > 0$ such that the function r from $(-\varepsilon + 1, 1+\varepsilon)$ into $\mathbf{R}$, given by $r(\alpha) = R(\alpha u)$, is strictly monotone decreasing.*

(vi) *There exists an even continuous function p from $H_0^1(D) - \{0\}$ into $(0, \infty)$ such that $R(p(u)u) = 0$ for all $u \in H_0^1(D) - \{0\}$. If $u \in \tilde{M}$, then $0 < p(u) \leq 1$.*

(vii) *For any $u \in \tilde{M}$ the function $g\colon \mathbf{R}_+ \to \mathbf{R}$, defined by $g(\alpha) = J(\alpha) - \frac{1}{2}R(\alpha u)$, is strictly monotone increasing.*

(viii) *There exists a constant S such that*

$$S < \int_D u(uf'(u) - f(u))\,dx < \infty$$

for all $u \in \tilde{M}$.

PROOF. (i): Let $u \in \tilde{M}$ be arbitrary. Then $u \neq 0$, and

$$0 \geq R(u) = \int_D |\nabla u|^2\,dx - \int_D uf(u)\,dx \geq \|u\|^2 - \varepsilon\|u\|_2^2 - B\|u\|_q^q.$$

Using Sobolev's imbedding theorem we obtain that

$$BK_1\|u\|^2((1-\varepsilon K_2)/BK_1 - \|u\|^{q-2}) \leq 0$$

for some $K_1, K_2 > 0$. Therefore

$$\|u\|^{q-2} \geq (1-\varepsilon K_2)/BK_1 > 0.$$

(ii): Let $u \in \tilde{M}$ be arbitrary. Then $R(u) \leq 0$. From part (i) and assumption (vi) we obtain that

$$\begin{aligned} J(u) - \frac{1}{2}R(u) &= \int_D \left(\frac{1}{2}uf(u) - F(u)\right)\,dx \\ &\geq \frac{p-1}{2(p+1)}\int_D uf(u)\,dx \geq \frac{p-1}{2(p+1)}\|u\|^2 > 0. \end{aligned}$$

Furthermore,

$$J(u) - \frac{1}{2}R(u) = \int_D \left(\frac{1}{2}uf(u) - F(u)\right)\,dx \leq \frac{p+3}{2(p+1)}\int_D uf(u)\,dx.$$

Since the imbedding $H_0^1(D) \hookrightarrow L^q(D)$ is compact, it follows that $J(u) - \frac{1}{2}R(u) < \infty$.

(iii): Let u be a nontrivial critical point of (1). Then, $-\Delta u - f(u) = 0$.

After multiplying this equation by u and integrating over D, we obtain that $R(u) = 0$; i.e., $u \in M$.

By assumptions (v) and (vi), R is C^2-differentiable. To show that M is a C^2-hypersurface, we need to show that $R'(u) \neq 0$ for all $u \in M$.

Suppose there exists a $u \in M$ such that $R'(u) = 0$. Then $2(-\Delta u - f(u)) = uf'(u) - f(u)$. After multiplying this by u and integrating over D, we obtain that

$$0 = 2R(u) = \int_D u(uf'(u) - f(u))\,dx.$$

Therefore, $u = 0$, which is impossible since $u \in M$. Hence, M is a C^2-submanifold of $H_0^1(D)$ of codimension 1.

(iv): Let $u \in M$ and $\alpha \in \mathbf{R}_+$ be arbitrary. Then,

$$j(\alpha) = \frac{1}{2}\alpha^2 \int_D |\nabla u|^2\,dx - \int_D F(\alpha u)\,dx,$$
$$j'(\alpha) = \alpha \int_D |\nabla u|^2\,dx - \int_D uf(\alpha u)\,dx,$$
$$j''(\alpha) = \int_D |\nabla u|^2\,dx - \int_D u^2 f'(\alpha u)\,dx = \int_D u(f(u) - uf'(\alpha u))\,dx.$$

Since f is strictly convex, it follows that $j'(\alpha) = 0$ if and only if $\alpha = 1$. Moreover, $j''(1) < 0$.

(v): Let $u \in \tilde{M}$ and $\alpha \in \mathbf{R}_+$ be arbitrary. Then,

$$r(\alpha) = \alpha^2 \int_D |\nabla u|^2\,dx - \int_D \alpha u f(\alpha u)\,dx,$$
$$r'(\alpha) = 2\alpha \int_D |\nabla u|^2\,dx - \int_D uf(\alpha u)\,dx - \int_D \alpha u^2 f'(\alpha u)\,dx.$$

Thus, since $R(u) \leq 0$ we obtain that

$$\begin{aligned} r'(1) &= 2\int_D |\nabla u|^2\,dx - \int_D uf(u)\,dx - \int_D u^2 f'(u)\,dx \\ &= 2R(u) + \int_D uf(u)\,dx - \int_D u^2 f'(u)\,dx < 0. \end{aligned}$$

Therefore, for α close enough to 1, $r(\alpha)$ is strictly monotone decreasing

(vi): Let $u \in H_0^1(D) - \{0\}$ and $\alpha \in \mathbf{R}_+$ be arbitrary. Then

$$\begin{aligned} R(\alpha u) &= \alpha^2 \int_D |\nabla u|^2\,dx - \int_D \alpha u f(\alpha u)\,dx \\ &\geq \alpha^2 \|u\|^2 - \varepsilon\alpha^2 \|u\|_2^2 - B\alpha^q \|u\|_q^q \\ &\geq \alpha^2 \|u\|^2 - \varepsilon\alpha^2 K_2 \|u\|^2 - B\alpha^q K_1 \|u\|^q \\ &= \alpha^2 B K_1 \|u\|^q \left(\frac{1 - \varepsilon K_2}{BK_1} \|u\|^{2-q} - \alpha^{q-2} \right) \end{aligned}$$

for some $K_1, K_2 > 0$. So, $R(\alpha u) > 0$ for α close to 0, and $R(\alpha u) < 0$ for α big enough. Hence, there exists an $\alpha_0 \in \mathbf{R}_+$ such that $R(\alpha_0 u) = 0$. Obviously, $\alpha_0 \in [0,1]$ if $u \in \tilde{M}$.

Uniqueness of α follows from the fact that for any $u \in H_0^1(D) - \{0\}$ the function s from $\mathbf{R}_+$ into $\mathbf{R}$ defined by

$$s(\alpha) = \frac{1}{\alpha} \int_D u f(\alpha u)\, dx$$

is strictly monotone.

All this implies that the function p is well defined. Moreover, it is an even function, since R is even. It remains to show continuity.

Let $u \in H_0^1(D) - \{0\}$. Let $p(u) = \alpha$. Since $R(\alpha u) = 0$, we obtain that

$$\alpha \int_D |\nabla u|^2\, dx = \int_D u f(\alpha u)\, dx$$

and

$$r'(\alpha) = \int_D u f(\alpha u)\, dx - \int_D \alpha u^2 f'(\alpha u)\, dx \neq 0.$$

The statement then follows from the Implicit Function Theorem.

(vii): Let $u \in \tilde{M}$ and $\alpha \in \mathbf{R}_+$ be arbitrary. Then,

$$g(\alpha) = \int_D \left(\frac{1}{2} \alpha u f(\alpha u) - F(\alpha u) \right) dx$$

and

$$\begin{aligned} g'(\alpha) &= \frac{1}{2} \int_D u f(\alpha u)\, dx + \frac{1}{2} \int_D \alpha u^2 f'(\alpha u)\, dx - \int_D u f(\alpha u)\, dx \\ &= \frac{1}{2} \int_D u(\alpha u f'(\alpha u) - f(\alpha u))\, dx > 0. \end{aligned}$$

(viii): The statement follows directly from assumptions (vi), (ii), and part (i).

This completes the proof of the proposition.

It is known that the tangent space of M at $u \in M$ can be identified with the following subspace of $H_0^1(D)$:

$$T_u M = \{v \in H_0^1(D) | d_u R(v) = \int_D R'(u) v\, dx = 0\},$$

which also has codimension one.

Let $u \in M$ be arbitrary but fixed. Then for any $w \in H_0^1(D)$ there is a unique decomposition $w = \alpha R'(u) + w^T$, where w^T is the projection of w into $T_u M$, and $\alpha = (w, R'(u))/\|R'(u)\|^2$.

Obviously, we can obtain a Finsler structure on M by assigning continuously to each point $u \in M$ the restriction of the inner product on $H_0^1(D)$ to $T_u M$.

2. Existence of infinitely many steady states. First we observe that the steady states of (1) are exactly the critical points of J. Since all of them are in M, it seems natural to apply the Ljusternik-Schnirelman category theory to $J_{|M}$ and to obtain infinitely many steady states (see [**3**, **4**]). In order to use the critical point theory we need the notion of genus.

DEFINITION 2.1. Let $\Sigma(M)$ be the set of all compact subsets of M, symmetric with respect to the origin. A set $A \in \Sigma(M)$ has genus k (denoted by $\gamma(A) = k$), if k is the smallest integer for which there exists an odd continuous map from A into $\mathbf{R}^k - \{0\}$. We set $\gamma(A) = \infty$, if there is no such finite k, and $\gamma(\emptyset) = 0$.

In the following proposition we state some properties of the genus. For the proofs see [**3**, **7**].

PROPOSITION 2.2. *Let $A, B \in \Sigma(M)$.*

(1) *If there exists an odd continuous map from A into B, then $\gamma(A) \leq \gamma(B)$.*

(2) *If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*

(3) *If there exists an odd homeomorphism from A into B, then $\gamma(A) = \gamma(B)$.*

(4) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.

(5) *If $\gamma(B) < \infty$, then $\gamma(\overline{A - B}) \geq \gamma(A) - \gamma(B)$.*

(6) *If A is compact, then $\gamma(A) < \infty$, and there exists a δ-neighborhood N of A such that $\gamma(N(A)) = \gamma(A)$.*

(7) *Let V be a k-dimensional subspace of $H_0^1(D)$ and $V^\perp$ its orthogonal complement. If $\gamma(A) > k$, then $A \cap V^\perp \neq \emptyset$.*

The proof of existence of infinitely many critical points of J in M is based on an application of the Deformation Lemma. We will state it in a form which is useful for our purposes. The proof is a modification of the proofs given in [**4**, **6**, **11**].

DEFINITION 2.3. We say that $J_{|M}$ satisfies the Palais-Smale condition if any sequences $\{u_n\}$ in M such that $\{J(u_n)\}$ is bounded, and $\|J'_{|M}(u_n)\| \to 0$, contains a subsequence which is convergent in M.

Let $A_b = \{u \in M | J(u) \leq b\}$ and $K_b = \{u \in M | J(u) = b \text{ and } J'_{|M}(u) = 0\}$ for all $b \in \mathbf{R}$.

Observe that if $J_{|M}$ satisfies the Palais-Smale condition, then K_b is compact.

LEMMA 2.4 (DEFORMATION LEMMA). *Suppose $J_{|M}$ satisfies the Palais-Smale condition. Let $b \in \mathbf{R}$ and U be any neighborhood of K_b. Then there exist constants $0 < \varepsilon < \bar{\varepsilon}$ and a map $\eta \in C(M, M)$ such that*

(i) *$\eta(u) = u$ for all $u \in M$ with $|J(u) - b| \geq \bar{\varepsilon}$;*

(ii) *η is an odd homeomorphism on M;*

(iii) *$J(\eta(u)) \leq J(u)$ for all $u \in M$;*

(iv) *$\eta(A_{b+\varepsilon} - U) \subset A_{b-\varepsilon}$;*

(v) *if $K_b = \emptyset$, then $\eta(A_{b+\varepsilon}) \subset A_{b-\varepsilon}$.*

The following theorem is standard, and shows existence of infinitely many steady states. Its proof can be found in [**3**].

THEOREM 2.5. *Let*

$$\Gamma_k = \{A \in \Sigma(M) | \gamma(A) \geq k\} \quad \text{and} \quad b_k = \inf_{A \in \Gamma_k} \sup_{u \in A} J(u).$$

Assume that $J_{|M}$ satisfies the Palais-Smale condition. Then b_k is a critical value of $J_{|M}$ for all $k \geq 1$, and $0 < \alpha \leq b_1 \leq \cdots \leq b_k \leq b_{k+1} \cdots$ for some $\alpha > 0$. If $b_m = b_{m+1} = \cdots = b_{m+r-1} = b$ for some $m, r \in \mathbf{N}$, then $\gamma(K_b) \geq r$.

In the following let $E_0 \subset E_1 \subset \cdots \subset E_s \subset \cdots$ be a sequence of finite-dimensional subspaces in $H_0^1(D)$ such that $\dim E_s = s$ and $\mathrm{cl}(\bigcup_{s=0}^\infty E_s) = H_0^1(D)$, where we denote by cl the closure in $H_0^1(D)$.

The following theorem shows the behavior of the critical values of $J_{|M}$. It also proves the existence of infinitely many critical points of J in M.

THEOREM 2.6. *$b_k \to \infty$ as $k \to \infty$.*

PROOF. Suppose $\{b_k\}$ is a bounded sequence in $\mathbf{R}$. Then there exists a constant $C > 0$ such that for any $k \in \mathbf{N}$ we can find a set $A_k \in \Gamma_k$ which satisfies the inequalities

$$0 < \sup_{u \in A_k \cap E_{k-1}^{\perp}} J(u) \leq \sup_{u \in A_k} J(u) \leq b_k + \frac{1}{k} < C.$$

Since $A_k \cap E_{k-1}^{\perp}$ is compact, there exists a point $u_k \in A_k \cap E_{k-1}^{\perp}$ such that

$$J(u_k) = \sup_{u \in A_k \cap E_{k-1}^{\perp}} J(u) < C.$$

In this way we obtain a sequence $\{u_k\}$ in $H_0^1(D)$ which is bounded by Proposition 1.1(ii). Therefore it contains a subsequence (denoted in the same way) which converges weakly in $H_0^1(D)$. Let u be its limit.

Lower semicontinuity of weak limits implies that $R(u) \leq \liminf_{k\to\infty} R(u_k) = 0$. Hence, $u \in \tilde{M}$, and therefore, by Proposition 1.1.(i), $u \neq 0$.

On the other hand, since $E_1^{\perp} \supset E_2^1 \supset \cdots \supset E_k^{\perp} \supset \cdots \supset \{0\}$, we obtain that $u_k \rightharpoonup 0$. Hence, $u = 0$, which is a contradiction. This proves the theorem.

Thus, in order to obtain infinitely many steady states, it is sufficient to show that $J_{|M}$ satisfies the Palais-Smale condition.

THEOREM 2.7. *Let $\{u_i\}$ be a sequence in M such that $\{J(u_i)\}$ is bounded and $\|J'_{|M}(u_i)\| \to 0$ as $i \to \infty$. Then $\{u_i\}$ contains a subsequence which converges to a critical point of J.*

PROOF. Let $\{u_i\}$ be a sequence in M such that $\{J(u_i)\}$ is bounded and $\|J'_{|M}(u_i)\| \to 0$ as $i \to \infty$. Proposition 1.1(ii) implies that $\{u_i\}$ is a bounded sequence in $H_0^1(D)$, and therefore it contains a weakly convergent subsequence (that we denote in the same way). Let u be its limit.

Since the imbedding $H_0^1(D) \hookrightarrow L^q(D)$ is compact, it follows that $u_i \to u$ in $L^q(D)$, and therefore

$$\lim_{i\to\infty} \int_D u_i f(u_i)\, dx = \int_D u f(u)\, dx$$

and

$$\lim_{i\to\infty} \int_D u^2 f'(u_i)\, dx = \int_D u^2 f'(u)\, dx.$$

Since $J'_{|M}(u_i) = J'(u_i) - \alpha_i R'(u_i)$ for some $\alpha_i \in \mathbf{R}$, we obtain that

$$(2\alpha_i - 1) \int_D (-\Delta u_i - f(u_i)) w\, dx - \alpha_i \int_D (u_i f'(u_i) - f(u_i)) w\, dx \to 0$$

as $i \to \infty$ for $w \in H_0^1(D)$.

In particular, it follows for $w = u_i$ that

$$(2\alpha_i - 1) R(u_i) - \alpha_i \int_D u_i (u_i f'(u_i) - f(u_i))\, dx \to 0$$

as $i \to \infty$. Therefore, since $R(u_i) = 0$, we obtain that $\alpha_i \to 0$ as $i \to \infty$. Hence, $-\Delta u_i - f(u_i) \to 0$ as $i \to 0$ in $H^{-1}(D)$. Therefore,

$$\int_D |\nabla u|^2\, dx = -\lim_{i\to\infty} \int_D u \Delta u_i\, dx = \lim_{i\to\infty} \int_D u f(u_i)\, dx = \int_D u f(u)\, dx.$$

Thus, $R(u) = 0$; i.e., $u \in M$.

On the other hand, we obtain that

$$\lim_{i\to\infty}\int_D |\nabla u_i|^2\,dx = \lim_{i\to\infty}\int_D u_i f(u_i)\,dx = \int_D uf(u)\,dx = \int_D |\nabla u|^2\,dx.$$

This implies that $u_i \to u$ in $H_0^1(D)$.

In particular, u is a critical point of J.

3. Instability of the steady states. Before discussing instability let us prove some preliminary propositions.

PROPOSITION 3.1. *Let $\Sigma(\tilde M)$ be the set of all compact subsets of $\tilde M$, symmetric with respect to the origin. Let $\tilde\Gamma_k = \{A \in \Sigma(\tilde M) | \gamma(A) \geq k\}$. Let*

$$c_k = \inf_{A\in\tilde\Gamma_k}\sup_{u\in A}\left(J(u) - \frac{1}{2}R(u)\right).$$

Then $b_k = c_k$.

PROOF. Since $\Gamma_k \subset \tilde\Gamma_k$, we obtain that $c_k \leq b_k$. So, it remains to show that $b_k \leq c_k$.

By definition of c_k, for any $\varepsilon > 0$ there is a set $A \in \tilde\Gamma_k$ such that

$$c_k \leq \sup_{u\in A}\left(J(u) - \frac{1}{2}R(u)\right) \leq c_k + \varepsilon.$$

Let $B = \{p(u)u | u \in A\}$, where p is the map defined in Proposition 1.1(vi). Then $B \in \Gamma_k$, and by Proposition 1.1(vi),(vii),

$$\begin{aligned} b_k &\leq \sup_{u\in B} J(u) = \sup_{u\in A}\left(J(p(u)u) - \frac{1}{2}R(p(u)u)\right) \\ &\leq \sup_{u\in A}\left(J(u) - \frac{1}{2}R(u)\right) \leq c_k + \varepsilon. \end{aligned}$$

Since ε was arbitrary, we obtain that $b_k \leq c_k$.

PROPOSITION 3.1. *Let ϕ be a critical point of $(J - \frac{1}{2}R)_{|\tilde M}$. Then $\phi \in M$ and ϕ is a critical point of J.*

PROOF. Suppose $\phi \in \tilde M - M$. Then

$$J'(\phi) - \tfrac{1}{2}R'(\phi) = \tfrac{1}{2}(\phi f'(\phi) - f(\phi)) = 0.$$

Since f is convex, it follows that $\phi = 0$. This contradicts Proposition 1.1(i). Hence, $\phi \in M$.

Since ϕ is a critical point of $(J - \frac{1}{2}R)_{|\tilde M}$, there exists a Lagrange multiplier $\lambda \in \mathbf{R}$ such that $J'(\phi) - (\frac{1}{2} + \lambda)R'(\phi) = 0$. Therefore,

$$2\lambda(-\Delta\phi - f(\phi)) = \left(\frac{1}{2} + \lambda\right)(\phi f'(\phi) - f(\phi)).$$

Multiplying the last equation by ϕ and integrating over D, we obtain that

$$0 = 2\lambda R(\phi) = \left(\frac{1}{2} + \lambda\right)\int_D \phi(\phi f'(\phi) - f(\phi))\,dx.$$

Since f is convex, it follows that $\lambda = -\frac{1}{2}$, and therefore, $J'(\phi) = 0$.

PROPOSITION 3.3. *The set*

$$\Delta = \{B \in \Sigma(\tilde{M}) | \gamma(A) \geq k,\ J(u) < b_k \text{ for all } u \in B\}$$

is not empty.

PROOF. By definition of b_k, for any $\varepsilon > 0$ there exists a set $A \in \Gamma_k$ such that $b_k \leq \sup_{u \in A} J(u) < b_k + \varepsilon$. From Proposition 1.1(viii) we obtain that

$$\int_D u(uf'(u) - f(u))\, dx > S$$

for all $u \in A$ and some $S > 0$.

Let $\alpha > 1$ but close to 1. Let $u \in A$ be arbitrary. Consider the function $j(\alpha) = J(\alpha u)$. Since $j'(1) = R(u) = 0$ and

$$j''(1) = \int_D u(f(u) - uf'(u))\, dx < -S,$$

we obtain that

$$J(\alpha u) < J(u) - \frac{1}{2} S(\alpha - 1)^2 < b_k + \varepsilon - \frac{1}{2} S(\alpha - 1)^2.$$

For ε small enough there exists an $\alpha > 1$ but close to 1 such that

$$(*) \qquad \varepsilon - \tfrac{1}{2} S(\alpha - 1)^2 \leq 0,$$

and therefore $J(\alpha u) < b_k$.

Consider the function $r(\alpha) = R(\alpha u)$, where α satisfies $(*)$. By Proposition 1.1(v), r is strictly monotone decreasing in a small neighborhood of 1, and therefore for ε small enough we obtain that $R(\alpha u) < 0$.

Let $B = \{\alpha u | u \in A\}$, where $\alpha > 1$ is such that $J(\alpha u) < b_k,\ R(\alpha u) < 0$ for all $u \in A$. Then $B \in \Delta$. This completes the proof of the proposition.

In the following we denote by $u(t, u^*)$ the solution of (1) with the following properties:

(i) $u(0, u^*) = u^*$;

(ii) $R(u(0, u^*)) < 0$;

(iii) $E(u(0, u^*), u_t(0, u^*)) < b_k$, where E is the functional, describing energy, defined in the following way:

$$E(u, v) = \frac{1}{2} \int_D |v|^2\, dx + \frac{1}{2} \int_D |\nabla u|^2\, dx - \int_D F(u)\, dx$$

for all $(u, v) \in H_0^1(D) \times L^2(D)$.

Moreover, we denote by $T(u^*)$ the time of existence of $u(t, u^*)$.

The following statement gives a condition for a solution of (1) to blow up in finite time; it was proved by L. E. Payne and D. H. Sattinger in [**13**].

PROPOSITION 3.4. *Let $u^* \in M$ be such that $R(u(t, u^*)) < 0$ for all $0 \leq t < T(u^*)$. Then $u(t, u^*)$ blows up in finite time.*

Thus, in order to prove instability of a steady state, it is sufficient to show that it can be approximated by a sequence $\{u_i^*\}$ such that $R(u(t, u_i^*)) < 0$ for all $0 \leq t < T(u_i^*)$ and all i. This is the idea of the following theorem.

THEOREM 3.5. *There exists an unstable critical point of* (1) *which corresponds to the critical value* b_k, $k \geq 2$. *In any neighborhood of this critical point there exists a solution of* (1) *which blows up in finite time.*

Let $\{A_i\}$ be a sequence in $\tilde{\Gamma}_k$ such that $J(u) < b_k$ and $-2/i < R(u) < 0$ for all $u \in A_i$. (Such sets can be obtained by the method used in Proposition 3.3.) Then

$$b_k \leq \sup_{w \in A_i} \left(J(w) - \frac{1}{2} R(w) \right) < b_k + \frac{1}{i}.$$

Let us denote by $u(t, u^*)$ the solution of (1) such that $u(0, u^*) = u^*$, and $u_t(0, u^*) = 0$. Let

$$\delta_i = b_k - \sup_{w \in A_i} J(w).$$

Then $R(w) > -3\delta_i$ for all $w \in A_i$. Furthermore, let

$$A_i^0 = \{w \in A_i | R(u(t, w)) > -\delta_i \text{ for some } t \geq 0\}$$

and

$$t(w) = \begin{cases} \inf\{t \geq 0 | -\delta_i < R(u(t, w)) \leq 0\}, & \text{for } w \in A_i^0, \\ \inf\{t \geq 0 | R(u(t, w)) < -3\delta_i\}, & \text{for } w \in A_i - A_i^0. \end{cases}$$

Furthermore, let

$$\tilde{d}(t, w) = \begin{cases} u(t, w), & \text{for } t < t(w), \\ u(t(w), w), & \text{for } t \geq t(w). \end{cases}$$

Since A_i is compact, it follows that

$$t^* = \sup\{t(w) | w \in A_i^0\} < \infty.$$

Let $d(w) = \tilde{d}(t^*, w)$. Then d is an odd continuous map from A_i into $H_0^1(D)$ for all i. Therefore, $B_i = d(A_i) \in \tilde{\Gamma}_k$.

Since energy is conserved under the flow of (1), it follows that $J(w) \leq b_k - \delta_i$ for all $w \in B_i$. Therefore,

$$J(w) - \tfrac{1}{2} R(w) < b_k - \delta_i + \tfrac{1}{2}\delta_i < b_k$$

for all $w \in d(A_i^0) \subset B_i$.

Let $\tilde{B}_i = \{v \in B_i | J(v) - \frac{1}{2}R(v) \geq b_k\}$. Proposition 3.1 implies that $\tilde{B}_i \neq \emptyset$. Moreover, from the above argument we obtain that $\tilde{B}_i \subset d(A_i - A_i^0)$; i.e., $u(t, v)$ blows up in finite time for all $v \in \tilde{B}_i$.

Now we claim that for any neighborhood U of K_{b_k} there exists an integer i such that $U \cap \tilde{B}_i \neq \emptyset$.

Let U be any neighborhood of K_{b_k}. Since f is strictly convex and $\tilde{M}$ is bounded away from zero, it follows that $(J - \frac{1}{2}R)_{|\tilde{M}}$ satisfies the Palais-Smale condition, and we can apply the Deformation Lemma (Lemma 2.4). Thus, there exists a homeomorphism $\eta \in C(\tilde{M}, \tilde{M})$ such that $\eta(A_{b_k+\varepsilon} - U) \subset A_{b_k-\varepsilon}$ for some ε small enough.

Since $J(v) - \frac{1}{2}R(v) < b_k + 2\delta_i$ for all $v \in B_i$, choosing i big enough we obtain that $\eta(B_i - U) \subset A_{b_k-\varepsilon}$. But, since $\eta(B_i) \subset \tilde{\Gamma}_k$, it follows that there exists a point $v_i \in U \cap B_i$ such that

$$b_k \leq J(\eta(v_i)) - \tfrac{1}{2}R(\eta(v_i)) \leq J(v_i) - \tfrac{1}{2}R(v_i).$$

Hence, $v_i \in U \cap \tilde{B}_i$.

In this way we obtain a sequence $\{v_i\}$ in $H_0^1(D)$ such that $v(t, v_i)$ blows up in finite time, $J(v_i) - \frac{1}{2}R(v_i) \to b_k$ as $i \to \infty$, the sequence $\{v_i\}$ converges to a critical point ϕ of $(J - \frac{1}{2}R)_{|\tilde{M}}$; i.e., ϕ is a critical point of J (see Proposition 3.2). This completes the proof of the theorem.

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